JOURNAL OF ALGEBRA **186,** 724–735 (1996)
ARTICLE NO. 0392

Ext and Frobenius, II

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Communicated by Melvin Hochster

In [D2] we proved the following theorem and derived several important results as applications.

THEOREM (1.5) [D2]. *Let (A, m, k) denote a complete* equidimensional *local ring of characteristic p (> 0)* without any imbedded component, *m its maximal ideal, and let $k = A/m$ be perfect. Denote by $F_\bullet = (F_i, d_i)_{i \geq 0}$ a complex of finitely generated free modules with finite length homologies. Write $F^n(F_\bullet)$ for $F_\bullet \otimes_A^{f^n} A$ where f^n is the Frobenius map repeated n-times $(f^n\colon A \to A,\ f^n(x) = x^{p^n})$. Let $w_{j,n}$ denote the jth cohomology of* $\operatorname{Hom}(F^n(F_\bullet), N)$ *and let $d =$* $\dim A$. *We have*

(i) *If* $\dim N < d$, $\lim l(w_{j,n})/p^{nd} = 0$.

(ii) *If* $\dim N = d$,

(a) *and if $j < d$*, $\lim l(w_{j,n})/p^{nd} = 0$,

(b) *while if $j = d$*, $\lim l(w_{j,n})/p^{nd} = \lim l(F^n(H_0(F_\bullet)) \otimes \tilde{N})/p^{nd}$.

(c) *If $j > d$ and* if $S^{-1}\tilde{N}$ is $S^{-1}A$ free of finite rank, $\lim l(w_{j,n})/p^{nd} = \lim l(H_{j-d}(F^n(F_\bullet)) \otimes \tilde{N})/p^{nd}$ *(here S denotes the set of all non-zero-divisors of A and $\tilde{N}$ stands for* $\operatorname{Hom}_A(H_m^d(N), E(k))$).

In this note our main goal is to prove the above theorem in the most general case (Theorem (1.5)) i.e., to prove the above theorem for *any* complete local ring A of ch $p\,(> 0)$ with perfect residue field; without requiring $S^{-1}\tilde{N}$ to be $S^{-1}A$ free of finite rank, i.e., N could be any finitely generated module over A. The method of proof requires a deeper digging

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0021-8693/96 $18.00

into spectral sequences in the light of the Frobenius map. The following result (Proposition 1.3), which I have been looking for a long time, plays a crucial role:

1.3. PROPOSITION. *Let $F_\bullet$ be as described in the statement of the above theorem. Then for any finitely generated module N,*

(i) *if* $\dim N < d$, $\lim l(\mathrm{Tor}_j(H_i(F^n(F_\bullet)), N))/p^{nd} = 0$ *for* $i, j \geq 0$,

(ii) *while if* $\dim N = d$, $\lim l(\mathrm{Tor}_j(H_i(F^n(F_\bullet)), N))/p^{nd} = 0$ *for* $j \geq 1$, $i \geq 0$.

As an immediate corollary we derive the following result

COROLLARY. *Let $F_\bullet$ and N be as above and assume* $\dim N = d$. *Then* $\lim l(H_i(F^n(F_\bullet) \otimes N))/p^{nd} = \lim l(H_i(F^n(F_\bullet)) \otimes N)/p^{nd}$ (*note that the left side* $= \lim l(H_i(F_\bullet \otimes^{f^n} N))/p^{nd}$).

Because of the more general nature of our main theorem (1.5), all the applications of the main theorem in [D2] can now be stated more generally. Since the proofs do not change at all, in this paper we mention them in Section 2 without providing any proof. We urge the reader to look up the corresponding proofs in [D2].

Notations. Throughout this work (A, m, k) will denote a complete local ring of dimension d (> 0) in positive characteristic p, m its maximal ideal, and $k = A/m$. We assume k is perfect. The Frobenius map f: $A \to A$, given by $f(x) = x^p$ for all $x \in A$, is a ring homomorphism. We denote by ${}^{f^n}A$ the bi-algebra A, having the structure of an A-algebra from the left by f^n and from the right by the identity map, i.e., $\alpha \in A$, $x \in {}^{f^n}A$, $\alpha x = \alpha^{p^n} x$, and $x \cdot \alpha = x\alpha$. $F_\bullet = (f_i, d_i)_{i \geq 0}$ will stand for a complex of finitely generated free modules with homologies of finite length. We write $F^n(F_\bullet)$ for $F_\bullet \otimes^{f^n} A$; given any A-module M, $F^n(M)$ for $M \otimes^{f^n} A$, and given any finitely generated module N, $\tilde{N}$ for $\mathrm{Hom}(H_m^d(N), E)$ where E denotes the injective hull of k over A. For any sequence $\{a_n\}$, $\lim a_n$ will denote the limit of $\{a_n\}$ as $n \to \infty$ (when it exists!). Unless mentioned, Tor and Ext are computed over A and A is dropped from the notations of both $\mathrm{Tor}_i^A(-, -)$ and $\mathrm{Ext}_A^i(-, -)$.

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1.1. LEMMA. *Let $D_n = (D_n^{r,s})_{r,s,n \geq 0}$ be a sequence of double complexes. Let us fix an integer t. We write $H_{t,n}$ to denote the tth homology of D_n and $E_{2,n}^{r,s}$ to denote the $E_2^{r,s}$ term of a spectral sequence* (*corresponding to horizontal or vertical filtration*) *of D_n converging to $H_{t,n}$. We assume $l(E_{2,n}^{r,s}) < \infty$ for all r, s, and n.*

(i) *Suppose that for r, s with $r + s = t$,* $\lim l(E_{2,n}^{r,s})/p^{nd} = 0$. *Then* $\lim l(H_{t,n})/p^{nd} = 0$.

(ii) *Suppose that for r, s with $r+s=t$ except for a particular value of s, say $s=s'$, all limits* $\lim l(E_{2,n}^{r,s})/p^{nd}=0$ *and* $\lim l(E_{2,n}^{r,s'})/p^{nd}>0$; *moreover suppose for $i\geq 2$ if $B_{i,s',n}$ denotes the image of $E_{i,n}^{r-i,s'+i-1}$ in $E_{i,n}^{r,s'}$ and $Q_{i,s',n}$ denotes the image of $E_{i,n}^{r,s'}$ in $E_{i,n}^{r+i,s'+i-1}$, that both* $\lim l(B_{i,s',n})/p^{nd}$ *and* $\lim l(Q_{i,s',n})/p^{nd}$ *are* 0. *Then*

$$\lim l(H_{t,n})/p^{nd}=\lim l\big(E_{2,n}^{t-s',s'}\big)/p^{nd}.$$

Proof. (i) Since $l(H_{t,n})=\sum_{r+s=t} l(E_{\infty,n}^{r,s})$ and $l(E_{\infty,n}^{r,s})\leq l(E_{2,n}^{r,s})$, the proof for this part is immediate.

(ii) Consider $0\to B_{2,s',n}\to E_{2,n}^{t-s',s'}\to Q_{2,s',n}\to 0$; the homology in the middle is $E_{3,n}^{t-s',s'}$. Since both $\lim l(B_{2,s',n})/p^{nd}$ and $\lim l(Q_{2,s'},n)/p^{nd}$ are 0, it follows that $\lim l(E_{3,n}^{t-s',s'})/p^{nd}=\lim l(E_{2,n}^{t-s',s'})/p^{nd}$. Continuing similar arguments for a finite number of steps, we conclude that $\lim l(E_{\infty,n}^{t-s',s'})p^{nd}=\lim l(E_{2,n}^{t-s',s'})/p^{nd}$. Now for $s\neq s'$, $r+s=t$, it is immediate that $\lim l(E_{\infty,n}^{r,s})/p^{nd}=0$.

As $l(H_{t,n})=\sum_{r+s=t} l(E_{\infty,n}^{r,s})$, we get the required conclusion from the above arguments.

1.2. Proposition. *Let M be a module of finite length and let N be any finitely generated module. We have*

(i) *If* $\dim N<d$, *then* $\lim l(\mathrm{Tor}_i(F^n(M),N))/p^{nd}=0$ *for* $i\geq 0$.

(ii) *If* $\dim N=d$, *then* $\lim l(\mathrm{Tor}_i(F^n(M),N))/p^{nd}=0$ *for* $i>0$.

Proof. (i) Let x be a parameter contained in $\mathrm{ann}_A N$. Because of the existence of a short exact sequence

$$0\to N'\to (A/xA)^s\to N\to 0$$

it is enough to show that $\lim l(\mathrm{Tor}_i(F^n(M),A/xA))/p^{nd}=0$. Write $\bar{A}=A/xA$, $\bar{M}=M/xM$. Observe that $\mathrm{Tor}_0(F^n(M),A/xA)=F^n_{\bar{A}}(\bar{M})$ and $l(\mathrm{Tor}_1(F^n(M),\bar{A}))\leq l((0:x)F^n(M))\leq l(F^n_{\bar{A}}(\bar{M}))$. Since $\dim\bar{A}=d-1$, $\lim l(F^n_{\bar{A}}(\bar{M}))/p^{nd}=0$ and hence $\lim l(\mathrm{Tor}_1(F^n(M),\bar{A}))/p^{nd}=0$. Assume $\lim l(\mathrm{Tor}_i(F^n(M),\bar{A}))/p^{nd}=0$ for $\forall i\leq j$. Then we observe that for any finitely generated A-module T such that $xT=0$, $\lim l(\mathrm{Tor}_i(F^n(M),T))/p^{nd}=0$ for $i\leq j$. This is obvious by applying $\otimes_A F^n(M)$ to the exact sequence

$$0\to T'\to (\bar{A})^n\to T\to 0$$

and considering the long exact sequence of Tor's.

Now consider the exact sequences

$$0\to (0:x)_A\to A\to xA\to 0$$

and

$$0 \to xA \to A \to A/xA \to 0.$$

It is clear that $\ell(\mathrm{Tor}_i(F^n(M), \bar{A})) = l(\mathrm{Tor}_{i-2}(F^n(M), (0:x)A))$ for $i > 2$ and $l(\mathrm{Tor}_2(F^n(M), \bar{A})) \le l(F^n(M) \otimes (0:x)A)$. Thus we are done, by the observation above.

(ii) Consider the exact sequence

$$0 \to N' \to A^s \to N \to 0.$$

We note, by a proposition in [S, Proposition 1, Sect. 3], that $l(\mathrm{Tor}_1(F^n(M), N))$ is a polynomial function of degree at most $(d-1)$. Hence $\lim l(\mathrm{Tor}_1(F^n(M), N))/p^{nd} = 0$. Thus, from the short exact sequence above, it follows that $\lim l(\mathrm{Tor}_i(F^n(M), N))/p^{nd} = 0$ for $i > 0$.

1.3. PROPOSITION. *Let $F_\bullet$ be as in the notation (i.e., a complex of finitely generated free modules with finite length homologies). Then for any finitely generated module N,*

(i) $\dim N < d, \lim l(\mathrm{Tor}_j(H_i(F^n(F_\bullet)), N))/p^{nd} = 0$ *for* $i, j \ge 0$ *and*

(ii) $\dim N = d, \lim l(\mathrm{Tor}_j(H_i(F^n(F_\bullet)), N))/p^{nd} = 0$ *for* $j \ge 1, i \ge 0$.

Proof. Write h_{in} to denote $H_i(F^n(F_\bullet))$.

(i) Choose a parameter $x \in \mathrm{ann}_A N$ and let $\bar{A} = A/xA$. Note that it is enough to show that $\lim l(\mathrm{Tor}_j(H_{in}, \bar{A}))/p^{nd} = 0$. Since $\dim \bar{A} = d-1$, we have, by Proposition 1.2, $\lim l(\mathrm{Tor}_i(H_{0n}, \bar{A}))/p^{nd} = 0$.

Now consider the double complex $D^n_{\bullet\bullet}$ obtained by tensoring $F^n(F_\bullet)$ with a free resolution $L_\bullet$ of $\bar{A}$ over A. We have the commutative diagram

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \to & & F^n(F_i) \otimes L_j & \to & F^n(F_{i-1}) \otimes L_j & & \to \\ & & \downarrow & & \downarrow & & \\ \to & & F^n(F_i) \otimes L_{j-1} & \to & F^n(F_{i-1}) \otimes L_{j-1} & & \to \\ & & \downarrow & & \downarrow & & \end{array} \tag{1}$$

Note that the vertical columns are all exact except at $j = 0$. Starting first with the horizontal rows we see that $E_2^{i,j} = \mathrm{Tor}_j(H_{in}, \bar{A})$.

We deal with H_{1n} first. Consider the exact sequence obtained from the spectral sequence above

$$\mathrm{Tor}_2\big(H_{0n}, \bar{A}\big) \to H_{1n} \otimes \bar{A} \to H_1\big(F^n(F_\bullet) \otimes \bar{A}\big) \to \mathrm{Tor}_1\big(H_{0n}, \bar{A}\big) \to 0.$$

This implies, by Proposition 1.2, that

$$\lim l\big(H_{1n} \otimes \bar{A}\big)/p^{nd} = \lim l\Big(H_1\big(F^n(F_\bullet) \otimes \bar{A}\big)\Big)/p^{nd} = 0,$$

since $\dim \bar{A} < d$.

Moreover, $l(\mathrm{Tor}_1(H_{1n}, \bar{A})) \le l((0:x)H_{1n}) \le l(H_{1n} \otimes \bar{A})$. Hence $\lim l(\mathrm{Tor}_1(H_{1n}, \bar{A}))p^{nd} = 0$. This implies that for all modules T such that $xT = 0$, $\lim l(\mathrm{Tor}_i(H_{1n}, T))/p^{nd} = 0$ for $i = 0, 1$. Consider the exact sequences

$$0 \to (0:x)_A \to A \to xA \to 0$$

and

$$0 \to xA \to A \to \bar{A} \to 0.$$

Since $\mathrm{Tor}_i(H_{1n}, \bar{A}) \simeq \mathrm{Tor}_{i-2}(H_{1n}, (0:x)_A)$ for $i > 2$ and $l(\mathrm{Tor}_2(H_{1n}, \bar{A})) \le l(H_{1n} \otimes (0:x)_A)$, we are done, by inducting on i.

Now suppose we have proved that for all $t \le i$, $\lim l(\mathrm{Tor}_j(H_{tn}, \bar{A}))/p^{nd} = 0$ for all $j \ge 0$. We will be done, by induction on i, if we can show that this implies that $\lim l(\mathrm{Tor}_j(H_{i+1,n}; \bar{A}))/p^{nd} = 0$ for all $j \ge 0$.

Note that it is enough to prove the above assertion for $j = 0$. For then, by repeating the process used in proving the case of H_{1n}, we will be done.

The $E_2^{p,q}$ terms which converge to $H_{i+1}(D^n_{\bullet\bullet}) = H_{i+1}(F^n(F_\bullet) \otimes \bar{A})$ are the following: $\mathrm{Tor}_0(H_{i+1,n}; \bar{A}), \mathrm{Tor}_1(H_{in}; \bar{A}), \ldots, \mathrm{Tor}_{i+1}(H_{0n}; \bar{A})$. By induction on i, the length of each, except the first one, in the above list tends to 0 in the limit. Also $\lim l(H_{i+1}(F^n(F_\bullet) \otimes \bar{A}))/p^{nd} = 0$ as noted earlier (a polynomial in n of degree $\le d-1$). Moreover in the spectral sequence, quotients of $E_2^{0,i+1}, E_3^{0,i+1}$, etc., will be formed modulo the images of submodules of $\mathrm{Tor}_2(H_{in}, \bar{A})$, $\mathrm{Tor}_3(H_{i-1,n}; \bar{A})$, etc., the length of each of which tends to 0 in the limit. Hence by Lemma 1.1, $\lim l(H_{i+1,n} \otimes \bar{A})/p^{nd} = 0$.

(ii) In Proposition 1.2, we already established that $\lim l(\mathrm{Tor}_j(H_{0n}, N))/p^{nd} = 0$ for $j \ge 1$. As in the case of (ii) of Proposition 1.2, it is enough to show that $\lim l(\mathrm{Tor}_1(H_{in}, N))/p^{nd} = 0$ $\forall i > 0$. Let us consider the double complex $D^n_{\bullet\bullet}$ obtained by tensoring $F^n(F_\bullet)$ with a minimal free resolution $L_\bullet$ of N. Similar diagrams and spectral sequence, as in (i) above, follow.

First we consider H_{1n}. We have the exact sequence

$$\to \mathrm{Tor}_2(H_{0n}, N) \to H_{1n} \otimes N \to H_1(F^n(F_\bullet) \otimes N) \to \mathrm{Tor}_1(H_{0n}, N) \to 0.$$

Since $\lim l(\mathrm{Tor}_2(H_{0n}, N))/p^{nd} = 0$ and $\lim l(\mathrm{Tor}_1(H_{0n}, N))/p^{nd} = 0$ (Proposition 1.2), we have

$$\lim l(H_{1n} \otimes N)/p^{nd} = \lim l(H_1(F^n(F_\bullet) \otimes N))/p^{nd}. \tag{1}$$

Now consider

$$0 \to N^1 \to A^s \to N \to 0. \tag{2}$$

Applying $\otimes H_{1n}$, we get

$$0 \to \operatorname{Tor}_1(H_{1n}, N) \to H_{1n} \otimes N^1 \to H_{1n} \otimes A^s \to H_{1n} \otimes N \to 0 \cdots . \quad (3)$$

By a proposition in [S, (1.6), Sect. 3] we know that

$$l\big(H_1\big(F^n(F_\bullet) \otimes A^s\big)\big) - l\big(H_1\big(F^n(F_\bullet) \otimes N\big)\big) - l\big(H_1\big(F^n(F_\bullet) \otimes N^1\big)\big)$$

is a polynomial in n of degree $\leq d-1$. Hence it follows from (1) and (3), that $\lim l(\operatorname{Tor}_1(H_{1n}, N))/p^{nd} = 0$.

Now suppose we have proved our result for H_{tn}, $\forall t \leq i$. We will now use induction on i, to prove the required result for $H_{i+1,n}$.

Note that $H_{i+1,n} \otimes N$, $\operatorname{Tor}_1(H_{in}, N)$, $\operatorname{Tor}_2(H_{i-1,n}; N)$, $\ldots$,$\operatorname{Tor}_{i+1}(H_{0n}, N)$ are the $E_2^{p,q}$ terms in the spectral sequence which converge to $H_{i+1}(D^n_{\bullet\bullet}) = H_{i+1}(F^n(F_\bullet) \otimes N)$. In this spectral sequence the quotient of $E_2^{0,i+1}$, $E_3^{0,i+1}$, etc., is formed by images of submodules of $\operatorname{Tor}_2(H_{in}, N)$, $\operatorname{Tor}_3(H_{i-1,n}; N)$, etc. Since the length of each of the $E_2^{p,q}$ terms, $p+q = i+1$, $p \neq 0$ and the length of each of $\operatorname{Tor}_s(H_{i-s+2}, N)$, after being divided by p^{nd} tend to 0 in the limit, we have, by Lemma 1.1,

$$\lim l(H_{i+1,n} \otimes N)/p^{nd} = \lim l\big(H_{i+1}\big(F^n(F_\bullet) \otimes N\big)\big)/p^{nd}. \quad (4)$$

Now from (2) we get

$$\begin{aligned} 0 \to \operatorname{Tor}_1(H_{i+1,n}, N) &\to H_{i+1,n} \otimes N^1 \to H_{i+1,n} \otimes A^s \\ &\to H_{i+1,n} \otimes N \to 0. \end{aligned} \quad (5)$$

By Proposition (1.6) of [S, Sect. 3], we know that $l(H_{i+1}(F^n(F_\bullet) \otimes A^s)) - l(H_{i+1}(F^n(F_\bullet) \otimes N^1)) - l(H_{i+1}(F^n(F_\bullet) \otimes N))$ is a polynomial in n of degree $\leq d-1$. Hence it follows, from (4) and (5) that $\lim l(\operatorname{Tor}_1(H_{i+1,n}, N))/p^{nd} = 0$.

Corollary (1). *Let $F_\bullet, N$ be as above and let* $\dim N = d$. *Then* $\lim l(H_i(F^n(F_\bullet) \otimes N))/p^{nd} = \lim l(H_i(F^n(F_\bullet)) \otimes N)/p^{nd}$ *for* $\forall i \geq 0$.

The proof follows directly from the proof of part (ii) of the above proposition.

Corollary (2). *Let P be a prime ideal of height i and let $x_1, \ldots, x_i$ be a part of a system of parameters contained in P. Then*

$$\begin{aligned} &\lim l\big(H_j\big(F^n(F_\bullet) \otimes A/(x_1, \ldots, x_i)\big) \otimes A/P\big)/p^{n(d-i)} \\ &\quad = \lim l\big(H_j\big(F^n(F_\bullet) \otimes A/P\big)\big)/p^{n(d-i)}. \end{aligned}$$

The proof is immediate from the complex $F_\bullet \otimes A/(x_1, \ldots, x_i)$ and the above corollary.

COROLLARY (3). *Let M be a module with $l(M) < \infty$, $pdM < \infty$, and P be as above. Then*

$$\lim l\Big(\mathrm{Tor}_j^A\big(F^n(M), A/x_1, \dots, x_i\big)\Big) \otimes A/P\big)/p^{n(d-i)}$$
$$= \lim l\Big(\mathrm{Tor}_j^A\big(F^n(M), A/P\big)\Big)/p^{n(d-i)}.$$

The proof is immediate from Corollary (2) above.

COROLLARY (4). *It makes sense to talk about $\chi_\infty(H_i, N)$. We define $\chi_\infty(H_i, N) = \sum_{j \geq 0}(-1)^j \lim l(\mathrm{Tor}_j(H_{in}, N))/p^{nd}$ and by Proposition* 1.3, $\chi_\infty(H_i, N) = \lim l(H_i(F^n(F_\bullet) \otimes N))/p^{nd}$.

Remark. When A is reduced, I have more direct proofs, without using Seibert's Proposition [S], for Proposition 1.2 and Proposition 1.3.

1.4. PROPOSITION. *Let $F_\bullet$ be as above and let N be any finitely generated module. Then we have the following*:

(i) *when* $\dim N < \dim A, \lim l(\mathrm{Ext}_A^j(H_i(F^n(F_\bullet)), N))/p^{nd} = 0$

(ii) *when* $\dim N = \dim A$ *and*

(a) $j < d$, *the above limit is* 0;

(b) $j = d, \lim l(\mathrm{Ext}_A^d(H_i(F^n(F_\bullet)), N))/p^{nd} = \lim l(H_i(F^n(F_\bullet)) \otimes \tilde{N})/p^{nd}$;

(c) $j > d, \lim l(\mathrm{Ext}^d(H_i(F^n(F_\bullet)), N))/p^{nd} = 0$.

Proof. We write T_n to denote $H_i(F^n(F_\bullet))$.
Let

$$I_\bullet : 0 \to I_0 \xrightarrow{\phi_0} I_1 \to \cdots \to I_{d-1} \xrightarrow{\phi_{d-1}} I_d \to \cdots$$

be a minimal injective resolution of N. Set $Z_j = \mathrm{Im}\ \phi_{j-1}$. We get an exact sequence

$$0 \to Z_{j-1} \to I_{j-1} \to Z_j \to 0. \tag{1}$$

Applying $\mathrm{Hom}_A(T_n, -)$, we get an exact sequence

$$0 \to \mathrm{Hom}\big(T_n, Z_{j-1}\big) \to \mathrm{Hom}\big(T_n, I_{j-1}\big) \to \mathrm{Hom}\big(T_n, Z_j\big)$$
$$\to \mathrm{Ext}^j\big(T_n, N\big) \to 0.$$

Writing $-^{v}$ for $\mathrm{Hom}(-, E)$, where E is the injective hull of k, we get from above

$$0 \to \mathrm{Ext}^j\big(T_n, N\big)^{v} \to T_n \otimes H_m^0\big(Z_j\big)^{v} \to T_n \otimes H_m^0\big(I_{j-1}\big)^{v}$$
$$\to T_n \otimes H_m^0\big(Z_{j-1}\big)^{v} \to 0. \tag{2}$$

Note that since T_n is a module of finite length, $\operatorname{Hom}(T_n, Z) = \operatorname{Hom}(T_n, H^0_m(Z_j))$ and since $H^0_m(Z_j)$ is a module with descending chain condition and A is complete $(H^0_m(Z_j)^\upsilon)^\upsilon$ is isomorphic to $H^0_m(Z_j)$ and thus

$$\operatorname{Hom}(T_n, Z_j) = \operatorname{Hom}\left(T_n \otimes H^0_m(Z_j)^\upsilon, E\right),$$

i.e., $\operatorname{Hom}(T_n, Z_j)^\upsilon = T_n \otimes H^0_m(Z_j)^\upsilon$. Similar arguments hold for I_j. Again from (1), applying $H^0_m(-)$, we get an exact sequence

$$0 \to H^0_m(Z_{j-1}) \to H^0_m(I_{j-1}) \to H^0_m(Z_j) \to H^j_m(N) \to 0.$$

Applying $\operatorname{Hom}(-, E)$ to the above sequence we obtain

$$0 \to H^j_m(N)^\upsilon \to H^0_m(Z_j)^\upsilon \to H^0_m(I_{j-1})^\upsilon \to H^0_m(Z_{j-1})^\upsilon \to 0.$$

We break up the sequence into two short exact sequences

$$0 \to H^j_m(N)^\upsilon \to H^0_m(Z_j)^\upsilon \to D \to 0;$$

$$0 \to D \to H^0_m(I_{j-1})^\upsilon \to H^0_m(Z_{j-1})^\upsilon \to 0.$$

Note that $H^0_m(I_{j-1})^\upsilon$ is a finitely generated free module and hence $\operatorname{Tor}^A_i(D, -) = \operatorname{Tor}^A_{i+1}(H^0_m(Z_{j-1})^\upsilon, -)$ for $i > 0$. Tensoring the above sequences with T_n, we get from (2), an exact sequence

$$\to \operatorname{Tor}^A_2\left(T_n, H^0_m(Z_{j-1})^\upsilon\right) \to H^j_m(N)^\upsilon \otimes T_n \to \operatorname{Ext}^j(T_n, N)^\upsilon$$
$$\to \operatorname{Tor}_1\left(T_n, H^0_m(Z_{j-1})^\upsilon\right) \to 0. \qquad (3)$$

(i) Since $\dim N < \dim A, \dim H^j_m(N)^\upsilon < \dim A$ the assertion follows from (3) by applying Proposition 1.3.

(ii) (a) $j < d$. The same argument as above completes the proof in this case.

(b) $j = d$. We get from (3) by Proposition 1.3,

$$\lim l\left(\operatorname{Ext}^d(T_n, N)\right)/p^{nd} = \lim l\left(T_n \otimes \tilde{N}\right)/p^{nd}.$$

(c) $j > d$. In this case $H^j_m(N)^\upsilon = 0$; thus (3) and Proposition 1.3 finish off the proof.

1.5. THEOREM. *Let $F_\bullet$ be as in the notations prior to Section 1. Let N be a finitely generated module. Let $W_{j,n}$ denote the jth homology of* $\mathrm{Hom}(F^n(F_\bullet), N)$. *We have the following*:

(i) *If* $\dim N < \dim A, \lim l(W_{j,n})/p^{nd} = 0$.

(ii) *If* $\dim N = \dim A$ *and*

(a) $j < d, \lim l(W_{j,n})/p^{nd} = 0$;

(b) $j = d, \lim l(W_{d,n})/p^{nd} = \lim l(F^n(H_0(F_\bullet)) \otimes \tilde{N})/p^{nd}$ *which is positive*;

(c) $j > d, \lim l(W_{j,n})/p^{nd} = \lim l(H_{j-d}(F^n(F_\bullet)) \otimes \tilde{N})/p^{nd}$.

Proof. We consider the double complex $D_n^{\bullet\bullet} = \mathrm{Hom}(F_\bullet, {}^{f^n}I_\bullet)$, where $I_\bullet$ is a minimal injective resolution of N. Let $V_{j,n}$ denote the jth homology of $D_n^{\bullet\bullet}$. Notice that $\mathrm{Hom}_A(F_r, {}^{f^n}I_s) = \mathrm{Hom}(F^n(F_r), I_s)$ and since F_r is free, $F^n(F_r)$ is also so.

We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\left(F_r, {}^{f^n}I_s\right) & \longrightarrow & \mathrm{Hom}\left(F_r, {}^{f^n}I_{s+1}\right) \\ \downarrow & & \downarrow \\ \mathrm{Hom}\left(F_{r+1}, {}^{f^n}I_s\right) & \longrightarrow & \mathrm{Hom}\left(F_{r+1}, {}^{f^n}I_{s+1}\right) \end{array}$$

Since $I_\bullet$ is exact except at $s = 0$, the horizontal rows of $D_n^{\bullet\bullet}$ are exact except at $s = 0$. Hence from the spectral sequence obtained by considering the horizontal rows first, we get $V_{j,n} = W_{j,n}$.

Now we consider the vertical rows first and the spectral sequence thus obtained has

$$E_{2,n}^{r,s} = \mathrm{Ext}^s\left(H_r\left(F^n(F_\bullet)\right), N\right).$$

The required conclusions now follow from Proposition 1.4.

Remark. When $F_\bullet$ is a free resolution of a module M of finite length, it can be easily shown from above arguments that for any finitely generated module N

$$\lim l(W_{d+1,n})/p^{nd} = \lim l\left(\mathrm{Tor}_1^A(M, {}^{f^n}A) \otimes \tilde{N}\right)/p^{nd}.$$

1.6. COROLLARY (to Theorem 1.5). *Let $F_\bullet$ be a free complex with finite length homologies and $I_\bullet = (I_n)_{n \geq 0}$ be a complex of modules with finitely generated homologies. We write $M = H_0(F_\bullet)$ and $N = H_0(I_\bullet)$, and denote the double complex* $\mathrm{Hom}(F_\bullet, {}^{fn}I_\bullet)$ *by $D_n^{\bullet\bullet}$. Let $W_{t,n}$ denote the tth homology of $D_n^{\bullet\bullet}$. Then*

$$\textit{for } t < d, \qquad \lim l(W_{t,n})/p^{nd} = 0$$

and

$$\text{for } t = d, \qquad \lim l(W_{d,n})/p^{nd} = \lim l(F^n(M) \otimes N)/p^{nd}.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\left(F_r, {}^{f^n} I_s\right) & \longrightarrow & \mathrm{Hom}\left(F_r, {}^{f^n} I_{s+1}\right) \\ \downarrow & & \downarrow \\ \mathrm{Hom}\left(F_{r+1}, {}^{f^n} I_s\right) & \longrightarrow & \mathrm{Hom}\left(F_{r+1}, {}^{f^n} I_{s+1}\right) \end{array}$$

We consider the spectral sequence obtained by taking the horizontal rows first and get

$$E_{2,n}^{r,s} = H_r\left(\mathrm{Hom}\left(F_\bullet, {}^{f^n} H_s(I_\bullet)\right)\right).$$

Now repeated applications of Theorem 1.5 complete the proof.

2

In this section we state several applications of Theorem (1.5). These are more general than the corresponding theorems in [D2]; but the proofs are identical almost to the word. Hence we do not provide any proof here; instead we request the reader to consult [D2].

2.1. THEOREM. *Let $F_\bullet$ be a finite free complex*:

$$0 \to A^{t_s} \to A^{t_{s-1}} \to \cdots \to A^{t_1} \to A^{t_0} \to 0.$$

Assume

(i) $l(H_i(F_\bullet)) < \infty$ *for* $i > 0$,

(ii) $H_m^0(H_0(F_\bullet)) \neq 0$, *and*

(iii) $H_0(F_\bullet)$ *is locally free on* spec $A - \{m\}$. *We define, for any finitely generated module N with depth* $N > 0$

$$\chi(F_\bullet, N) = l\left(H_m^0(F_\bullet \otimes N)\right) - l\left(H_1(F_\bullet \otimes N)\right) + l\left(H_2(F_\bullet \otimes N)\right) \cdots$$

and

$$\chi_\infty(F_\bullet, N) = \lim \chi\left(F^n(F_\bullet), N\right)/p^{nd}.$$

If $s < d$, $\chi_\infty(F_\bullet, N) = 0$; *if* $s = d$ *and* $\dim N = d$, $\chi_\infty(F_\bullet, N) > 0$.

Proof. See the proof of Theorem (1.7) in [D2].

Remark. The above theorem establishes the following observation: Let M be a finitely generated module, locally free on spec $A - \{m\}$ with depth $M = 0$. Let $F_\bullet$ be as in the above theorem with $s < d$ and $H_0(F_\bullet) \simeq M$. Then there cannot exist a minimal free complex $L_\bullet$ of magnitude d, $l(H_i(L_\bullet)) < \infty$ for $i > 0$, such that $H_0(L_\bullet) \simeq M$.

2.2. *Let $F_\bullet$ be a finite free complex with finite length homologies and let the magnitude of $F_\bullet$ be $s + d$. We impose the following grading on $F_\bullet^* =$* $\mathrm{Hom}_A(F_\bullet, A)$: $(F_\bullet^*)_i = (F_{s+d-i})^*$. *In the next theorem we are going to find out the relation between $\chi_\infty(F_\bullet, N)$ and $\chi_\infty(F_\bullet^*, N)$.*

THEOREM. *Let $F_\bullet$ be as above and let N be a finitely generated module. Then*

(i) *if* $\dim N < d$, $\chi_\infty(F_\bullet, N) = 0$

(ii) *if* $\dim N = d$, *then* $\chi_\infty(F_\bullet, N) = (-1)^s \chi_\infty(F_\bullet^*, N)$.

Proof. See the proof of Proposition (1.8) in [D2].

2.3. *Now we furnish a proof of the improved new intersection conjecture in characteristic $p\,(> 0)$ by using Theorem* 2.1 *above.*

THEOREM. *Let (A, m, k) be as in the Notations. Let $F_\bullet$ be a finite free complex $0 \to F_s \to F_{s-1} \to \cdots \to F_1 \to F_0 \to 0$ such that $l(H_i(F_\bullet)) < \infty$ for $i > 0$ and $H_0(F_\bullet)$ has a minimal generator (non-zero) killed by a power of m. Then* $\dim A \leq s$.

Proof. See the proof of Theorem (2.2) in [D2].

2.4. *Remark.* The Remark at the end of Theorem 2.1 can be extended to prove the following result:

THEOREM. *Let (A, m, k) be as above. Let $F_\bullet = (F_i)_{i \geq 0}$ and $G_\bullet = (G_i)_{i \geq 0}$ be two finite free complexes of magnitudes r and s, respectively, such that*

(i) $r \neq s$, $r, s \leq d$,

(ii) $l(H_i(F_\bullet)) < \infty$, $l(H_i(G_\bullet)) < \infty$ *for $i > 0$, and*

(iii) $H_0(F_\bullet)$ *and* $H_0(G_\bullet)$ *are locally free on* spec $A - \{m\}$. *Then* $H_0(F_\bullet)$ *is* not *isomorphic to* $H_0(G_\bullet)$.

The author can prove the above result in the mixed characteristic case when the mixed characteristic $p\,(> 0)$ is a non-zero-divisor on A and both $r, s \leq d - 1$.

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